International Journal of Theoretical Physics, Vol. 25, No. 10, 1986

New Framework for the Feynman Path Integral

Shaharir b. M. Z[1]

Received May 20, 1986

The well-known Fourier integral solution of the free diffusion equation in an arbitrary Euclidean space is reduced to Feynmannian integrals using the method partly contained in the formulation of the Fresnelian integral. By replacing the standard Hilbert space underlying the present mathematical formulation of the Feynman path integral by a new Hilbert space, the space of classical paths on the tangent bundle to the Euclidean space (and more general to an arbitrary Riemannian manifold) equipped with a natural inner product, we show that our Feynmannian integral is in better agreement with the qualitative features of the original Feynman path integral than the previous formulations of the integral.

1. INTRODUCTION

Since Feynman (1948) postulated his path integral, numerous authors have proposed various mathematical formulations to establish the existence and provide an understanding of the vague integral. The most notable formulations are by Kac (1949, 1959), De Witt (1957), Cameron (1960, 1962-63, 1968), Ito (1961), Nelson (1964), Cheng (1972, 1973), DeWitt-Morette (1972, 1974, 1976, 1979), McLughlin and Schulman (1971), Albevario and Hoegh-Krohn (1976, 1979), Truman (1976, 1977, 1978, 1979), DeWitt-Morette et al. (1979, 1980), Tarski (1979), Elworthy and Truman (1981), and Streit and Hida (1983). A recent review which contains extensive bibliographies on this subject prior to 1980 and focusses on Euclidean spaces is found in Exner (1985).

Despite this extensive work on its foundation and also on its application [the latter can be seen in Schulman (1981) and Glimm and Jaffe (1981)], the Feynman path integral still needs to be examined and elaborated, at least in the following aspects:

1. A clearer link between the various mathematical formulations listed above with the original Feynman path integral (Feynman 1948;

[1]Pusat Pengajian Kuantitatif, Universiti Kebangsaan Malaysia, Bangi, Selangor, Malaysia.

0020-7748/86/1000-1075$05.00/0

Feynman and Hibbs, 1965) and the usual integrals needs to be exposed.

2. Relationships between the various formulations of the integral, such as the one produced by Johnson (1982) and Kun Soo Chang et al. (1984), are yet to be fully established.
3. We still need a better deliberation on the agreement, or otherwise, of path integrals on Riemannian manifolds, particularly the formulations by DeWitt-Morette (1979), DeWitt-Morette et al. (1979, 1980), and Elworthy and Truman (1981) with other quantization procedures, namely that of Schwinger's action integral (Cohen and Shaharir, 1974), the Born–Jordan–Dirac procedure (Lin et al., 1970), or even with the Feynman path integrals formulated by DeWitt (1957), Cheng (1972, 1973), and McLughlin and Schulman (1971), and with the stochastic mechanics as discussed by Shaharir (1986b).

In this paper we give yet another framework for formulating the Feynman path integral, essentially aiming toward generalizing the work of Albevario and Hoegh-Krohn (1967, 1979) and producing a path integral on Riemannian manifolds that contains all the qualitative features of the original Feynman path integral, especially the concept of "sum over *all classical* paths."

First, we carefully derive a Feynmannian integral from the well-known solution of the free diffusion equation (the heat equation without external source or the Shrödinger equation for a free particle) in $\mathbb{R}^n$. This is done via different Hilbert spaces: a refinement of the traditional space

$$H = \{\gamma \in L_0^{2,1}([0, t]; \mathbb{R}^n), \text{with } \gamma(t) = 0 \\ \text{and inner product } m/\hbar \int_0^t \dot{\gamma}_1(s) \cdot \dot{\gamma}_2(s)\, ds\}$$

into

$$H_{m/\hbar}^{\text{cl}} = \{\text{classical paths on } \mathbb{R}^n \text{ but with the same inner product as in } H\}$$

and the generalization of $H_{m/\hbar}^{\text{cl}}$,

$$\chi_{\|\gamma}(P_{\text{cl}}) = \{\text{vector fields parallel to classical path } \gamma \\ \text{with inner product } m/\hbar \int_0^t g(x(s), y(s))\, ds\}$$

Even though other authors, particularly Albevario and Hoegh-Krohn (1976, 1979), have essentially used the same mathematical basis (but limited to the Fourier integral and the algebra of complex-bounded Borel measures on $\mathbb{R}^n$ and H) in formulating their celebrated Fresnelian integral, and Elworthy and Truman (1981) have done their analysis on the set of vector fields, but with more general elements than our elements in $\chi_{\|\gamma}(P_{\text{cl}})$ and different inner product, I feel that the present framework differs from theirs in the following respects:

1. The present framework is more natural, especially when considering our proposal to base the Feynman path integral on the manifold of classical paths whose tangent space is restricted to $\chi_{\parallel\gamma}(P_{cl})$ even for the case of diffusion in Euclidean space $\mathbb{R}^n$. This has been overlooked by previous authors. [The inner product used by Elworthy and Truman (1981) is very much different from ours, and for our purposes we find that their inner product is not suitable for formulating our Feynmannian path integral.]
2. Our approach establishes very clearly the link between our path integral with the original Feynman path integral and ordinary integrals.
3. This paper provides a new framework for formulating a Feynmannian path integral on Riemannian manifolds in a very much different setting than those formulated by DeWitt-Morette (1979), DeWitt-Morette et al. (1979, 1980), and Elworthy and Truman (1981), but generalizing the Fresnelian integral of Albevario and Hoegh-Krohn (1976, 1979) and Truman (1978).

2. A DERIVATION OF A FEYNMANNIAN PATH INTEGRAL ON EUCLIDEAN SPACE

It is well known (for example, Reed and Simon, 1979) that by applying the method of Fourier integral calculus to the Euclidean free diffusion equation (heat equation without external source or the Schrödinger equation for free particle in $\mathbb{R}^n$)

$$k\frac{\partial\psi}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2\psi}{\partial x_i^2}, \qquad \operatorname{Re} k \in \mathbb{R}^+ \tag{1}$$

$$\psi(x, 0) = \phi(x) \tag{2}$$

one obtains the solution

$$\psi(x, t) = (4\pi t/k)^{-n/2} \int_{\mathbb{R}^n} [\exp(-|z|^2 k/4t)]\phi(x+z)\, dz \tag{3}$$

We will show that for a class of ϕ, this solution can be reduced to a standard form of the Feynman path integral,

$$\psi(x, t) = \int \exp\left[\frac{i}{\hbar}\int_0^t L(\dot{x}, x)\, ds\right] dx \tag{4}$$

where the integral is prescribed as "the sum over all classical paths" corresponding to the classical Lagrangian L of a free particle with mass m,

$$L(\dot{x}, x) = \tfrac{1}{2}m \sum_{j=1}^{n} \dot{x}_j^2, \qquad \dot{x} \equiv dx/dt, \tag{5}$$

and $\hbar$ is the Planck constant. Our aim is to produce a precise meaning for this integral which contains all of the qualitative prescriptions of the integral described by Feynman. Theorem 2.0 below provides a partial solution to the problem. Theorem 2.0 depends on the following lemmas.

Lemma 2.1. Let

$$L^{2,\beta}([0,t);\mathbb{R}^n)=\{\gamma\colon[0,t]\to\mathbb{R}^n,\ \gamma \text{ has derivatives up to } \beta \text{ order a.e., and each derivative is square-integrable in } [0,t]\}$$

$$H^{\mathrm{cl}}_{m/\hbar}=\{\gamma\in L^{2,2}([0,t];\mathbb{R}^n),\ \ddot{\gamma}=0 \text{ a.e., and for any } \gamma \text{ and } \eta \text{ of such functions } \langle\gamma|\eta\rangle=m/\hbar\textstyle\int_0^t\dot{\gamma}(s)\cdot\dot{\eta}(s)\,ds\}$$

Then $H^{\mathrm{cl}}_{m/\hbar}$ is a separable Hilbert space (an overdot denotes the derivative d/ds, and a centered dot is the Euclidean inner product).

Proof. The fact that $H^{\mathrm{cl}}_{m/\hbar}$ is a vector space with inner product $\langle\cdot|\cdot\rangle$ is trivial and both are classical results. Now we show that every limit point y of a sequence $\{y_k\}\in H^{\mathrm{cl}}_{m/\hbar}$ is in the set itself. We have $y_k(s)=a_ks+b_k$; $a_k, b_k\in\mathbb{R}^n$; and

$$\langle y-y_k|y-y_k\rangle=\frac{m}{\hbar}\int_0^t(\dot{y}-a_k)\cdot(\dot{y}-a_k)\,ds$$

which shows that

$$\dot{y} \text{ is a limit point for } \{a_k\} \text{ in } L^2$$

and by completeness of L^2,

$$\dot{y}=a\in\mathbb{R}^n \text{ a.e., where } a \text{ is limit point of } \{a_n\}$$

and hence $\ddot{h}=0$ a.e.

$H^{\mathrm{cl}}_{m/\hbar}$ is separable since $\{id, \mathbf{1}\colon id(s)=s,\ \mathbf{1}(s)=1\}$ is a countable base of $H^{\mathrm{cl}}_{m/\hbar}$. ■

Lemma 2.2. (1) Let $C[(0,t);\mathbb{R}^n]$ be a set of continuously differentiable maps from the interval $(0,t)$ into the n-Euclideam space whose standard inner product is denoted by g: $g(a,b)=a\cdot b$; and let

$$ev_s\colon C[(0,t)\colon\mathbb{R}^n]\to\mathbb{R}^n$$

$$ev_s(f)=f(s)$$

the evaluation map at s. Then $\langle,\rangle$, an inner product on $C[(0,t);\mathbb{R}^n]$ defined by

$$\langle\gamma,\lambda\rangle=g(\gamma(s),\lambda(s))=\gamma(s)\cdot\lambda(s)$$

is the pullback of g by ev_s,

$$\langle,\rangle=ev_s^*(g)$$

2. Let $C'[(0, t); \mathbb{R}^n]$ be a set of continuously differentiable maps from $(0, t)$ into $\mathbb{R}^n$. Then $\langle , \rangle$, an inner product on $C'[(0, t); \mathbb{R}^n]$ defined by

$$\langle \gamma, \lambda \rangle = \dot{\gamma}(s) \cdot \dot{\lambda}(s)$$

is the pullback of g by a map

$$dev_s : C'[(0, t); \mathbb{R}^n] \to \mathbb{R}^n$$

$$dev_s(f) = \dot{f}(s)$$

3. The inner product on $H^{\text{cl}}_{m/\hbar}$ defined in Lemma 2.1 is a pullback of the Euclidean metric g by the map

$$dev_{s_0,\alpha} : H^{\text{cl}}_{m/\hbar} \to \mathbb{R}^n, \qquad \alpha = \left(\frac{tm}{\hbar}\right)^{1/2}, \qquad t > 0$$

$$dev_{s_0,\alpha}(\lambda) = \alpha \dot{\lambda}(s_0), \qquad s_0 \text{ is fixed}$$

and the inner product g is a pullback of $\langle , \rangle$, the metric on $H^{\text{cl}}_{m/\hbar}$ by the map

$$F : \mathbb{R}^n \to H^{\text{cl}}_{m/\hbar}$$

$$F(y) = \lambda, \qquad \lambda(s) = (t - s) y (\hbar / mt)^{1/2}$$

Proof. (1) Since ev_s is linear, then, by the definition of the Frechet derivatives D,

$$D(ev_s)(f_0)(g) = g(s) \tag{6}$$

for any $f_0 \in C[(0, t) : \mathbb{R}^n]$ and $g \in T_{f_0} C[(0, t); \mathbb{R}^n]$, the tangent space of $C[(0, t); \mathbb{R}^n]$ at f_0, which, in this case, may be identified by $C[(0, t); \mathbb{R}^n]$ itself [see Abraham et al. (1983) for a more general evaluation map and its derivatives, and Golubitsky and Guillemin (1973) for manifolds of mappings]. By the definition of the pullback (*) (for example, Lang, 1985; Abraham et al., 1983)

$$\begin{aligned}
\langle \lambda, \gamma \rangle &= ev_s^*(g)(\lambda, \gamma), \qquad \lambda, \gamma \in T_{\lambda_0} C[(0, t); \mathbb{R}^n] \\
&= g(ev_s(\lambda_0))(T_{\lambda_0} ev_s(\lambda), T_{\lambda_0} ev_s(\gamma)) \\
&= g(\lambda_0(s))(\lambda(s), \gamma(s)), \qquad \text{by (6)} \\
&= \lambda(s) \cdot \gamma(s), \qquad \text{definition}
\end{aligned}$$

(2) Since dev_s is linear, once again we have

$$\begin{aligned}
T_{f_0}(dev_s)(g) &= (dev_s(f_0), D(dev_s)(f_0)(g)) \\
&= (f_0(s), \dot{g}(s))
\end{aligned}$$

The result is obtained through the same argument as in 1.

(3) Since $dev_{s_0,\alpha}$ is linear,

$$T_{\lambda_0}\, dev_{s_0,\alpha}(\lambda) = (dev_{s_0,\alpha}(\lambda_0),\, \alpha\dot{\lambda}(s_0))$$

By the result in 2,

$$\begin{aligned}\langle \lambda, \gamma\rangle &= \alpha^2\dot{\lambda}(s_0)\cdot\dot{\gamma}(s_0),\\ &= \frac{\alpha^2}{t}\int_0^t \dot{\lambda}(s)\cdot\dot{\gamma}(s)\, ds\end{aligned}$$

since $\dot{\lambda}(s)\cdot\dot{\lambda}(s)$ is constant for any s. Hence, the result is obtained as asserted. For the second part,

$$F(y) = (\mathbf{t} - id)y\left(\frac{\hbar}{mt}\right)^{1/2}, \qquad id(s) = s, \qquad \mathbf{t}(s) = t$$

which is linear in y. Thus

$$DF(y_0)(y) = (\mathbf{t} - id)y(\hbar/mt)^{1/2}$$

and hence

$$\begin{aligned} g(y_0)(y, z) &= F^*(\langle,\rangle)(y, z)\\ &= \langle T_{y_0}F(y),\, T_{y_0}F(z)\rangle\\ &= (\hbar/mt)\langle(\mathbf{t} - id)y, (\mathbf{t} - id)z\rangle\\ &= (1/t)\int_0^t (-y)\cdot(-z)\, ds\\ &= y\cdot z \quad \blacksquare\end{aligned}$$

We can now prove our first theorem on the reduction of a class of solutions to the diffusion equation into Feynmannian path integrals.

Theorem 2.0. Suppose $\phi:\mathbb{R}^n \to \mathbb{C}$ is Lebesque square-integrable and a Fourier transform of a complex bounded Borel measure μ on $\mathbb{R}^n$; then the solution of the Euclidean free diffusion equation (3) can be reduced to a Feynmannian path integral

$$\psi(x, t) = \int_{H^{\text{cl}}_{m/\hbar}} \exp[i/\hbar \textstyle\int_0^t L(q(s), \dot{q}(s))\, ds]\, dD_F(q) \tag{7}$$

where

$$L = e^{i(\pi/2 - p)}\mathscr{L}, \qquad p = \text{phase of } k$$

$$\mathscr{L}(q(s), \dot{q}(s)) = \tfrac{1}{2}m\dot{q}(s)\cdot\dot{q}(s)$$

is the classical Lagrangian for a free particle with mass m;

$H^{\text{cl}}_{m/\hbar} = \{q \in L^{2,2}([0, t]\colon \mathbb{R}^n),\ \ddot{q} = 0$ a.e., and for any q_i of such functions $\langle q_1 | q_2\rangle = m/\hbar \int_0^t \dot{q}_1(s)\cdot\dot{q}_2(s)\, ds$ is well defined$\}$

is the space of classical paths for a free particle; and a Feynmannian measure D_F is given by

$$dD_F = [T_{\lambda_0}(dev_{s_0,\alpha})]^*(d\mu_{\phi,x,s})$$

$$\lambda_0 \in H^{\text{cl}}_{m/\hbar}$$

$$dev_{s_0,\alpha} : H^{\text{cl}}_{m/\hbar} \to \mathbb{R}^n$$

$$\lambda \mapsto \alpha\lambda(s_0), \qquad \alpha = (mt/\hbar)^{1/2}$$

$$s_0 \text{ is arbitrary}$$

$$d\mu_{\phi,x,s} = (S^{-1})^*(d\mu_{\phi,x})$$

$$S : \mathbb{R}^n \to \mathbb{R}^n$$

$$z \mapsto y = (|k|/2t)^{1/2} z, \qquad \text{a scaling map}$$

$$d\mu_{\phi,x}(y) = e^{ix\cdot y}\, d\mu_\phi(y)$$

or equivalently

$$D_F(A) = \int_{(F\circ T)^{-1}(A)} e^{ix\cdot y}\, d\mu_\phi(y)$$

$$F : \mathbb{R}^n \to H^{\text{cl}}_{m/\hbar}$$

$$y \mapsto \lambda, \qquad \lambda(s) = (t-s)y(\hbar/mt)^{1/2}$$

$$T = S^{-1} : \mathbb{R}^n \to \mathbb{R}^n, \qquad T(y) = y(2t/|k|)^{1/2}$$

Proof. We have, for Re $k > 0$, from (3),

$$\begin{aligned}
\psi(x, t) &= \int_{\mathbb{R}^n} (4\pi t/k)^{-n/2}\, e^{z\cdot zk/4t} \phi(x+z)\, dz \\
&= \int_{\mathbb{R}^n} (4\pi t/k)^{-n/2}\, e^{-z\cdot zk/4t} \\
&\quad \times \left[\int_{R^n} e^{-(z+x)\cdot y}\, d\mu_\phi(y) \right] dz \qquad \text{by assumption} \\
&= \int_{\mathbb{R}^n} e^{-y\cdot yt/k}\, e^{ix\cdot y}\, d\mu_\phi(y)
\end{aligned}$$

by Fubini's theorem and the properties of the "normal distribution" [with some reservations, as discussed by Shaharir (1986)a)]. Hence,

$$\psi(x,t)=\int_{\mathbb{R}^n} e^{-y\cdot yt/k}\, d\mu_{\phi,x}(y)$$

$[d\mu_{\phi,x}=e^{ix\cdot y}\, d\mu_\phi(y)$

a complex bounded Borel measure (e.g., Exner, 1985)]

$$=\int_{\mathbb{R}^n} \exp[-\exp(-ip)z\cdot z/2]\, d\mu_{\phi,x,s}(z)$$

$[p=\text{phase of } k, \qquad \mu_{\phi,x,S}=\mu_{\phi,x}\circ S^{-1},$

$S^{-1}:\mathbb{R}^n\to\mathbb{R}^n, \qquad y\mapsto z=y(2t/|k|)^{1/2}]$

or in terms of differential forms

$$d\mu_{\phi,x,s}=(S^{-1})^*(d\mu_{\phi,x})$$

the pullback of $d\mu_{\phi,x}$ by S^{-1}. [Association of a measure with a differential form is well known; see, for example, Abraham *et al.* (1983).]

Now

$$\psi(x,t)=\int_{T_bR^n} \exp[aG_{\lambda_0}(\lambda)\cdot G_{\lambda_0}(\lambda)]\, d\mu_{\phi,x,S}(G_{\lambda_0}(\lambda))$$

$$a=\tfrac{1}{2}e^{-ip}$$

$$b=\alpha\lambda_0(s_0), \qquad \lambda_0\in H^{\text{cl}}_{m/\hbar}$$

Choose

$$G_{\lambda_0}(\lambda)=T_{\lambda_0}f(\lambda)$$

where $f=dev_{s_0,\alpha}$ is defined as in the theorem. Then by Lemma 2.2, part 3, and the general theory of the transformation variable of an integral of a differential form, we have

$$\psi(x,t)=\int_{T_{\lambda_0}H^{\text{cl}}_{m/\hbar}} e^{-a\langle\lambda|\lambda\rangle}\, dD_F(\lambda)$$

In fact, by identifying $d\mu_{\phi,x,S}$ as ω, a differential form, we have

$$G^*_{\lambda_0}(\exp\circ\|\ \|^2_{\mathbb{R}^n}\cdot\omega)$$

$$=(\exp\circ\|\ \|^2_{\mathbb{R}^n})\circ G_{\lambda_0}\cdot G^*_{\lambda_0}(\omega)$$

$$=\exp\circ\|\ \|^2_{\mathbb{R}^n}\circ G_{\lambda_0}\cdot G^*_{\lambda_0}(\omega)$$

$$=\exp\circ(\|\ \|^2_{\mathbb{R}^n}\circ G_{\lambda_0})\cdot G^*_{\lambda_0}(\omega)$$

$$=\exp\circ g(T_{\lambda_0}f, T_{\lambda_0}f)\cdot G^*_{\lambda_0}(\omega), \qquad g \text{ is the } \cdot \text{ product,}$$

$$=\exp\circ\|\ \|^2_{H^{\text{cl}}_{m/\hbar}}\cdot G^*_{\lambda_0}(\omega) \qquad \text{by Lemma 2.2, part 3}$$

The result is obtained by identifying $G^*_{\lambda_0}(\omega)$ as a measure; $T_{\lambda_0}H^{\text{cl}}_{m/\hbar}$ as $H^{\text{cl}}_{m/\hbar}$; and

$$-1 = i^2 = i\exp(i\pi/2)$$

The other parts of the theorem are proved by similar arguments, but using the second part of Lemma 2.2, part 3, and the theory of Borel measures, since all the relevant transformations T and F are proper maps. The measure D_F is obtained by using the Radon-Nikodym theorem. ■

Remark 2.0. We note that Theorem 2.0 is of exactly the same form as the original Feynman path integral (Feynman, 1948; Feynman and Hibbs, 1965) for the case of the Schrödinger equation, $k=-2mi/\hbar$, *including* "the sum over all the classical paths."

Our method differs significantly from that contained in the formalism of the Fresnelian integral of Albevario and Hoegh-Krohn (1976, 1979), especially the important step regarding the role of "the reproducing kernel" (see also Exner, 1985) for transforming an integral

$$\int_{\mathbb{R}^n} e^{ix\cdot y}\, d\mu(y)$$

into an integral on the well-known Hilbert space

$$H = \{\gamma \in L^{2,1}([0, t]; \mathbb{R}^n), \gamma(t) = 0,$$
$$\langle \gamma, \lambda\rangle = \textstyle\int_0^t \dot\gamma(s)\cdot\dot\lambda(s)\, ds\}$$

of the form

$$\int_H e^{i\langle\gamma,\lambda\rangle}\, d\nu(\lambda)$$

by letting $x=\gamma(0)$ and $y \mapsto \lambda$, $\lambda(s) = (t-s)y$. Lemma 2.2, part 3, and also the proof of Theorem 2.0 show that their arguments are not completely justifiable.

It is natural to seek a better result than Theorem 2.0. Theorem 2.1 below shows that, with a slightly different argument, *all* solutions of the free diffusion (not necessarily just the Schrödinger equation) can in fact be reduced to exactly the same form as the original Feynman path integral.

Theorem 2.1. Suppose the initial distribution of the free diffusion problem is such that

$$m_x(A) = \int_A \exp\{\exp[-(z-x)\cdot(z-x)\alpha]\}\phi(z)\, dz, \qquad \operatorname{Re}\alpha > 0$$

for all x, defines a complex-bounded Borel measure m_x on $\mathbb{R}^n$; then the solution of the free diffusion problem can be reduced to a Feynmannian path integral in exactly the form of the original Feynmann path integral,

$$\psi(x, t) = \int_{H_{m/\hbar}^{\mathrm{cl}}} \exp\left[i/\hbar \int_0^t L(\dot{q}(s), q(s))\, ds \right] dD_F(q)$$

where L is the classical Lagrangian for a free particle in $\mathbb{R}^n$ and

$$D_F(B) = \int_{F^{-1}(B)} \left(\frac{4\pi t}{k}\right)^{-n/2} e^{-z\cdot z\alpha} \phi(x+z)\, dz,$$

$$\alpha = \frac{k}{4t} - \frac{i}{2}, \qquad \operatorname{Re} k > 0$$

where F is given in Theorem 2.0. $(m_x(A))$ is just an expected value of $\phi(z)$,

$$z \sim N\left(x, \frac{1}{2\sqrt{\alpha}} I_{n\times n}\right), \qquad z \in A$$

Proof. We have

$$\psi(x, t) = \int_{\mathbb{R}^n} (4\pi t/k)^{-n/2}\, e^{z\cdot zk/t} \phi(z+x)\, dz$$

$$= \int_{\mathbb{R}^n} (4\pi t/k)^{-n/2}\, e^{iz\cdot z/2}\, e^{-z\cdot z(k/4t+i/2)} \phi(z+x)\, dz$$

However, given

$$m_x(A) = \int_A e^{-|z-x|^2\alpha} \phi(z)\, dz, \qquad \operatorname{Re} \alpha > 0$$

$$= \int_{T^{-1}(A)} e^{-y\cdot y\alpha} \phi(x+y)\, dy$$

$$= m_{x,T}(A), \qquad T(z) = z - x = y$$

is a complex-bounded Borel measure, so that $m_{x,T}$ is a complex-bounded Borel measure on $\mathbb{R}^n$, since T is clearly a homemorphism; hence

$$\psi(x, t) = \int_{\mathbb{R}^n} e^{iz\cdot z/2}\, d\nu_{x,t}(z)$$

$$d\nu_{x,t}(z) = (4\pi t/k)^{-n/2}\, e^{-z\cdot z(k/4t-i/2)} \phi(x+z)\, dz$$

is a complex-bounded Borel measure.

Finally, use the transformation $F: z \mapsto H_{m/\hbar}^{\mathrm{cl}}$ of G given in the proof of Theorem 2.0 to obtain the result.

The following corollary is not surprising, since it is essentially the result implicitly contained in the Fresnelian integral of Albevario and Hoegh-Krohn (1976, 1979).

Corollary 2.1. If ϕ is a Fourier transform of a complex-bounded Borel measure μ_ϕ on $\mathbb{R}^n$, then Theorem 2.1 holds.

Proof. N_x defined by

$$N_x(B) = \int_B e^{-(z-x)\cdot(z-x)\alpha}\, dz$$

is a complex-bounded Borel measure in $\mathbb{R}^n$ for all $x \in \mathbb{R}^n$ and $\operatorname{Re} \alpha > 0$: it is in fact a "complex" normal distribution with mean x and variance $(1/2\sqrt{\alpha})I_{n\times n}$ (Shaharir, 1986a). Thus,

$$\begin{aligned} m_x(A) &= \int_A e^{-(z-x)\cdot(z-x)\alpha} \left[\int_{\mathbb{R}^n} e^{iz\cdot y}\, d\mu_\phi(y) \right] dz \\ &= \int_A \left[\int_{\mathbb{R}^n} e^{iz\cdot y}\, d\mu_\phi(y) \right] dN_x(z) \\ &\le |N_x|(A) |\mu_\phi|(\mathbb{R}^n) < \infty \end{aligned}$$

which shows that m_x satisfies the hypothesis of Theorem 2.1.

Remark 2.1. Even though we have obtained our objective of deriving a Feynmannian path integral in Euclidean space for all free diffusion through Theorem 2.1, it is clear that the framework used to obtain the result is not suitable for a generalization, because in general space, classical paths (even for free diffusion) do not constitute a vector space and the expression

$$\int_0^t \dot{\gamma}(s) \cdot \dot{\eta}(s)\, ds \qquad \text{for curves } \gamma \text{ and } \eta$$

on a manifold is meaningless.

Thus, we need an entirely new framework to overcome this probelm, such as the one proposed by Elworthy and Truman (1981), Elworthy (1982), and DeWitt-Morette et al. (1979, 1980), and DeWitt-Morette (1979). However, we have obtained a different formulation (and different results) than those mentioned above, as shown in the following section.

3. A DERIVATION OF A FEYNMANNIAN PATH INTEGRAL ON RIEMANNIAN MANIFOLDS

In this section we propose to generalize our main results in the previous section, so that a Feynmannian path integral for diffusion on Riemannian manifolds can be obtained in a very similar way to that in the Euclidean space formulated in Theorem 2.1.

First we prove several lemmas, which generalize the lemmas in the previous section and provide the basic framework for Theorem 3.1.

Lemma 3.1. Let M be a Riemannian manifold with a metric g. Then:

1. A curve $\gamma \in C^2([0, t]; M)$ on M is a geodesic iff $g(\dot\gamma, \dot\gamma)$ is constant and $\dot\gamma$ is not orthogonal to $\nabla_{\dot\gamma}\dot\gamma$, $\dot\gamma \neq 0$.
2. The set of all geodesics on a Riemannian manifold is a submanifold of all C^2 paths on the manifold.

Proof. (1) If γ is a geodesic, then $g(\dot\gamma, \dot\gamma)$ is constant (see, e.g., Spivak, 1979) and by definition, $\dot\gamma$ is not orthogonal to $\nabla_{\dot\gamma}\dot\gamma$. If $g(\dot\gamma, \dot\gamma)$ is constant, then

$$\frac{d}{ds} g(\dot\gamma, \dot\gamma) = 0$$

which implies

$$g(\dot\gamma, \nabla_{\dot\gamma}\dot\gamma) = 0$$

and hence

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

if $\dot\gamma \neq 0$ or $\dot\gamma$ is not orthogonal to $\nabla_{\dot\gamma}\dot\gamma$.

2. Consider

$$F: C'[I; M] \to C[I; \mathbb{R}]$$

where $C'[I; M]$ is a set of C'-smooth paths on M, and

$$F(\gamma) = g(\dot\gamma, \dot\gamma)$$

We will show that F is $C^\infty$and any $r > 0$ is a regular value of F, so that by the well-known theorem on submersion (see, e.g., Abraham et al., 1983) $F^{-1}(r)$ is a submanifold, which is the required result, by the first part of this lemma.

The smoothness of F follows from the smoothness of g.

First we show that $T_\gamma F(\eta) = (F(\gamma), 2g(\gamma, \nabla_{\dot\gamma}\eta))$. This can be shown by reinterpreting the classical expression δg for $g(\dot x, \dot x) = g_{jk}(x)\dot x^j \dot x^k$, which is given by

$$\delta g = 2g_{jm}\dot x^j(\delta\dot x^m + \Gamma^m_{kl}\dot x^k \delta\dot x^l)$$

A rigorous proof can be formulated by a slight modification of the modern formulation of the global calculus of variation (see, e.g., Choquet-Bruhart et al., 1982) as follows.

We have

$$F = \hat{g} \circ \tilde{\gamma}, \qquad \tilde{\gamma}: C'[I; M] \to C(I; TM)$$

$$\tilde{\gamma}(\gamma) = (\gamma, \dot{\gamma}) \qquad \text{a kind of lift of } \gamma$$

$$\hat{g}: C(I; TM) \to C[I; \mathbb{R}]$$

$$\hat{g}(\tilde{\gamma}) = g \circ \tilde{\gamma}$$

$T_\gamma F(\eta)$ is in

$$\frac{d}{du} F(C_u)\big|_{u=0} = T\hat{g} \circ \frac{d\tilde{C}_u}{du}\bigg|_{u=0}$$

$$\frac{d\tilde{C}_u}{du}: \{u\} \times I \to C(I; TTM)$$

$$C_{(\cdot)}: (-\varepsilon, \varepsilon) \to C'[I; M], \quad C_0 = \gamma$$

Now

$$T\hat{g}: C(I; TTM) \to C(I; T\mathbb{R})$$

$$T\hat{g}(\tilde{\tilde{x}}): C(I; T_{\tilde{x}}TM) \to C(I; T_{\hat{g}(\tilde{x})}\mathbb{R})$$

which shows that the first component of $T_\gamma F(\eta)$ is correct. However, to obtain the full expression of $T_\gamma F(\eta)$, we follow the standard argument in calculating $(d/du)F(C_u)$ in a coordinate of M, (q, U), and the coordinate of TM, $((q, \dot{q}), TU)$, to obtain

$$\frac{d}{du} F(C_u) = (\hat{g}, D\hat{g}) \circ (\tilde{C}_u, D\tilde{C}_u)_{(q,\dot{q})}$$

$$= (\hat{g} \circ \tilde{C}_u, D\hat{g} \circ \tilde{C}_u \cdot D\tilde{C}_u)_{(s,d/ds)}$$

But in this coordinate

$$D\hat{g} \circ \tilde{C}_u \cdot D\hat{C}_u = \frac{\partial \hat{g}}{\partial q^j} \circ \hat{C}_u \cdot \frac{\partial C_u^j}{\partial u} + \frac{\partial \hat{g}}{\partial \dot{q}^j} \circ \hat{C}_u \frac{\partial^2 C_u^j}{\partial u\, \partial s}$$

$$\left[\text{since } \tilde{C}_u = \left(C_u, \frac{\partial C_u}{\partial s}\right)\right]$$

$$= \frac{\partial g_{kl}}{\partial q^i} \frac{\partial C_u^k}{\partial s} \frac{\partial C_u^l}{\partial s} \frac{\partial C_u^i}{\partial u} + 2 g_{kl} \frac{\partial C_u^k}{\partial s} \frac{\partial}{\partial s}\left(\frac{\partial C_u^l}{\partial u}\right)$$

which is the result by identifying $\eta = (\partial C_u / \partial u)\big|_{u=0}$ and the relation between the Christoffel symbols and the derivative of the Riemannian metric.

Now, given any $f \in C(I; T\mathbb{R})$, there exists $\eta \in C'[I; TM]$ such that $T_\gamma F(\eta) = f$, since

$$g(\dot{\gamma}, \nabla_{\dot{\gamma}}\eta) = f_2, \qquad f = (f_1, f_2)$$

is a linear ordinary differential equation in η. This shows that $T_\gamma F$ is surjective at γ, where $F(\gamma) = f_1$, and $\dot{\gamma} \neq 0$. $Ker(T_\gamma F)$ splits $T_\gamma C$, since it consists of η such that

$$g(\dot{\gamma}, \nabla_{\dot{\gamma}}\eta) = 0$$

whose solution is a vector space spanned by a finite number of basis solutions.

By definition, any $f \in C(I; \mathbb{R}^+)$ is a regular value of F. ■

Lemma 3.2 Let

$$\chi_{\|q}(P_{\rm cl}) = \{X: I = [0, t] \to TM, X(s) \in T_{q(s)}M, q \in P_{\rm cl}, \nabla_{\dot{q}}X = 0, \text{and}$$
$$\langle X \mid Y \rangle = m/\hbar \textstyle\int_0^t g(X, Y)\, ds \text{ is well defined}\}$$

Then $\chi_{\|q}(P_{\rm cl})$ is a closed subspace of $T_qP_{\rm cl}$, the tangent space to the manifold of the classical paths $P_{\rm cl}$ at q.

Proof. It is clear that $\chi_{\|q}(P_{\rm cl})$ is a vector space, since $\nabla_{\dot{q}}$ is linear and $\langle \cdot | \cdot \rangle$ is obviouisly an inner product on $\chi_{\|q}(P_{\rm cl})$, since g is an inner product. By definition, it is also obvious that $\chi_{\|q}$ is a subset of $T_qP_{\rm cl}$ (e.g., Eliasson, 1967), so we only need to show that any limit point is in $\chi_{\|q}(P_{\rm cl})$.

By a well-known result regarding vector fields along a geodesic (e.g., Spivak 1979), we may assume

$$x = c\dot{q}, \qquad c \in \mathbb{R}^n.$$

Suppose $X = f\dot{q}$ is the limit of an arbitrary sequence $\{X_k\}$ in $\chi_{\|q}(P_{\rm cl})$. Then

$$\|X - X_k\|^2 = \frac{m}{\hbar}\int_0^t g((f - c_k)\dot{q}, (f - c_k)\dot{q})\, ds$$
$$= \frac{m}{\hbar}\int_0^t (f - c_k)^2 \|\dot{q}\|_M^2\, ds$$

which shows that f is the limit of $\{c_k\}$ in $L^2_{\|\dot{q}\|_M^2}(I; \mathbb{R})$, the space of Labesque square-integrable functions with weight $\|\dot{q}\|_M^2$. Thus f is constant almost everywhere and hence X is in $\chi_{\|q}(P_{\rm cl})$. ■

Using Lemmas 3.1 and 3.2, we now propose the following theorem, which generalizes the results in the previous section.

Theorem 3.1. Let M be a proper Riemannian manifold with metric $g: TM \times TM \to \mathbb{R}$,

$$P_{\text{cl}} = \{q: [0, t] \to M,\ q \text{ is a } C^2\text{-classical path of a free particle on } M\}$$

$$\chi_{\|q}(P_{\text{cl}}) = \{\boldsymbol{\lambda}: [0, t] \to TM, \boldsymbol{\lambda}(s) \in T_{q(s)}M, \quad q \in P_{\text{cl}}, \nabla_{\dot q}\boldsymbol{\lambda} = 0,$$
$$\boldsymbol{\lambda} \in L_g^2([0, t]; TM) \text{ in the sense that for any other } \boldsymbol{\gamma} \text{ of such a map } \langle \boldsymbol{\lambda} | \boldsymbol{\gamma} \rangle = m/\hbar \textstyle\int_0^t g(\boldsymbol{\lambda}(s), \boldsymbol{\gamma}(s))\, ds\}$$

1. The expression

$$\phi(X) = \int_{T_{m_0}M} e^{ig(m_0)(X,Y)}\, d\mu(Y)$$

where μ is a complex bounded Borel measure on T_mM, can be reduced to

$$\phi(X) = \int_{\chi_{\|q}(P_{\text{cl}})} \exp(i/\hbar < \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle\, d\nu(\boldsymbol{\beta})$$

where

$$X = k\boldsymbol{\alpha}(s_0), \qquad s_0 \text{ is arbitrary}$$
$$k^2 = mt/\hbar$$
$$d\nu = F^*(d\mu)$$
$$F: \chi_{\|q}(P_{\text{cl}}) \to T_{m_0}M, \qquad m_0 = q(s_0)$$
$$\boldsymbol{\lambda} \mapsto k\boldsymbol{\lambda}(s_0)$$

2. Any function $\psi: TM \times I \to \mathbb{R}$,

$$\psi(x, t) = \int_{T_{m_0}M} e^{-\beta(t)g(z,z)} N(\beta(t), t)\phi(x+z)\, d\mu_{T_{m_0}M}(z)$$

where $\mu_{T_{m_0}M}$ is the volume element of $T_{m_0}M$, m_0 is an arbitrary element of M, $\operatorname{Re}\beta(t) > 0$, N is a normalizing factor, and $\phi: T_{M_0}M \to \mathbb{C}$ is such that

$$\nu(A) = \int_A e^{-\alpha(t)g(z-x,z-x)}\phi(z)\, d\mu_{T_{m_0}M}(z)$$

defines a complex bounded Borel measure on $T_{m_0}M$ for all X, $\operatorname{Re}\alpha(t) > 0$, can be reduced to a Feynmannian integral

$$\psi(X, t) = \int_{\chi_{\|q}(P_{\text{cl}})} \exp\left[i/\hbar \int_0^t L(\boldsymbol{\alpha}(s))\, ds \right] dD_F(\boldsymbol{\alpha})$$

where

$$L(\boldsymbol{\alpha}(s)) = \tfrac{1}{2}mg(\boldsymbol{\alpha}(s), \boldsymbol{\alpha}(s)), \qquad \boldsymbol{\alpha} \in \chi_{\|q}(P_{\text{cl}})$$

In particular, this theorem holds if ϕ satisfies part 1 of this theorem.

Proof. (1) We may identify the integrand as a differential form Ω_X on $T_{m_0}M$,

$$\Omega_X = \exp \circ ig(X, \cdot)\, d\mu_{T_{m_0}M}$$

For any smooth $F: \chi_{\|q}(P_{\text{cl}}) \to T_{m_0}M$ we have

$$\begin{aligned} F^*\Omega_X &= (\exp \circ ig(X, \cdot)) \circ F \cdot F^*(d\mu) \\ &= \exp \circ (ig(X, \cdot) \circ F) \cdot F^*(d\mu) \end{aligned}$$

Choose $F(\boldsymbol{\alpha}) = k\boldsymbol{\alpha}(s_0)$ for some fixed $s_0 \in [0, t]$; we have

$$\begin{aligned} F^*\Omega_X(\boldsymbol{\beta}) &= \exp[ig(X, k\boldsymbol{\beta}(s_0))]F^*(d\mu)(\boldsymbol{\beta}) \\ &= \exp[ig(k\boldsymbol{\alpha}(s_0), k\boldsymbol{\beta}(s_0))]F^*(d\mu)(\boldsymbol{\beta}), \qquad X = k\boldsymbol{\alpha}(s_0) \end{aligned}$$

But

$$\begin{aligned} \frac{d}{ds}(g(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= g(\nabla_{\dot q}\boldsymbol{\alpha}, \boldsymbol{\beta}) + g(\boldsymbol{\alpha}, \nabla_{\dot q}\boldsymbol{\beta}) \\ &= 0 \qquad \text{by hypothesis} \end{aligned}$$

Thus

$$g(\boldsymbol{\alpha}(s_0), \boldsymbol{\beta}(s_0)) = \frac{1}{t}\int_0^t g(\boldsymbol{\alpha}(s), \boldsymbol{\beta}(s))\, ds \quad \text{for any} \quad s_0 \in [0, t]$$

Hence,

$$\begin{aligned} F^*\Omega_X(\boldsymbol{\beta}) &= \exp\left[i(k^2/t)\int_0^t g(\boldsymbol{\alpha}(s), \boldsymbol{\beta}(s))\, ds\right]F^*(d\mu)(\boldsymbol{\beta}) \\ &= \exp(i/\hbar\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle)F^*(d\mu)(\boldsymbol{\beta}) \end{aligned}$$

where

$$\begin{aligned} \langle\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle &= \frac{k^2}{t}\int_0^t g(\boldsymbol{\alpha}(s), \boldsymbol{\beta}(s))\, ds \\ &= \frac{m}{\hbar}\int_0^t g(\boldsymbol{\alpha}(s), \boldsymbol{\beta}(s))\, ds, \qquad k^2 = \frac{mt}{\hbar} \end{aligned}$$

This completes the proof for the first part of the theorem.

(2) We have

$$\psi(X, t)=\int_{T_mM} \exp[i/2g(Z, Z)-g(Z, Z)(\beta+i/2)] \times N(\beta, t)\phi(X+Z)\, d\mu_{T_mM}(Z)$$

But

$$\nu(A)=\int_A \exp[-\alpha g(Z-X, Z-X)]\phi(Z)\, d\mu_{T_mM}(Z)$$

$$=\int_{\tau^{-1}(A)} \exp[-\alpha g(Y, Y)]\phi(X+Y)\, d\mu_{T_mM,\tau}(X)$$

[where $\tau(Z)=Z-X=Y$] is given as a complex bounded Borel measure; hence, we may write

$$\psi(X, t)=\int_{T_mM} e^{i/2g(Z,Z)}\, d\nu_{X,t}(Z)$$

where $\nu_{X,t}$ is a complex bounded Borel measure on T_mM given by

$$d\nu_{X,t}(Z)=N(\beta, t)\, e^{-(\beta+i/2)g(Z,Z)}\phi(X+Z)\, d\mu_{T_mM}(Z)$$

By the same argument as in step 1 above,

$$\psi(X, t)=\int_{\chi\|_\gamma(P_{cl})} \exp(i/2\langle\boldsymbol{\alpha},\boldsymbol{\alpha}\rangle)\, d\nu(\boldsymbol{\alpha})$$

where

$$d\nu=F^*(d\nu_{X,t})$$

$$F\colon \chi\|_\gamma(P_{cl})\to T_mM, \qquad m=\gamma(s_0)$$

$$F(\boldsymbol{\alpha})=\left(\frac{mt}{\hbar}\right)^{1/2}\boldsymbol{\alpha}(s_0)$$

$$\tfrac{1}{2}\langle\boldsymbol{\alpha}|\boldsymbol{\alpha}\rangle=\frac{m}{2\hbar}\int_0^t g(\boldsymbol{\alpha}(s), \boldsymbol{\alpha}(s))\, ds=\frac{1}{\hbar}\int_0^t L(\boldsymbol{\alpha}(s))\, ds$$

$$L(\boldsymbol{\alpha}(s))=\tfrac{1}{2}mg(\boldsymbol{\alpha}(s), \boldsymbol{\alpha}(s))$$

The second part is proved by the same argument as in the corollary of Theorem 2.1. ■

Theorem 3.1 shows that "the sum over all classical paths" can only be realized (in the case of diffusion on a Riemannian manifold) in terms of "the sum over all vector fields parallel to the classical paths" and the classical Lagrangian involved is given (in a natural coordinate system) by

$$L(x, \dot{x}) = \tfrac{1}{2} m g_{jk}(x) \dot{x}^j \dot{x}^k$$

where x is the integral curve for the vector field parallel to the classical path.

Theorem 3.1 already suggests that a Fourier integral operator, analogous to the operator discussed in great detail by Hormander (1971), of the form

$$F_s(\phi)(X) = \int_{T_xM} e^{iS(X,Y)} \phi(Y) \, d\mu_{T_xM}(Y)$$

where

$$S(X, Y) = g(X, Y) + iS_2(X, Y)$$

for some S_2 would be the most suitable operator for solving the diffusion equation on a Riemannian manifold. The solution then can be reduced to a Feynmannian path integral (using the results here) for a class of initial distribution ϕ such that

$$\phi(X) = \int_{T_xM} e^{iS(X,Y)} \, d\nu_\phi(Y)$$

for a complex bounded Borel measure ν_ϕ on T_xM associated with ϕ. Accordingly, a Fresnelian integral on Riemannian manifolds can also be formulated. I will discuss these aspects elsewhere.

ACKNOWLEDGMENTS

This work was done while the author was on sabbatical leave at the Department of Mathematics, University of California, Santa Cruz, California, as a Fulbright Scholar 1985/86, with Prof. Ralph Abraham as Faculty Associate.

The author wishes to express his gratitude to the UKM for granting sabbatical leave, MACEE for a Fulbright Fellowship, Prof. Ralph Abraham for his support, hospitality, and resourcefulness, and Dotty Hollinger for

her patience and professional work in turning the author's scribbles into this manuscript.

REFERENCES

Abraham, R., Marsden, J. E., and Ratiu, T. (1983). *Manifolds, Tensor Analysis and Applications*, Addison-Wesley, Reading, Massachusetts.

Albevario, S. A., and Hoegh-Krohn, R. J. (1976). *Mathematical Theory of Feynman Path Integrals*, Lecture Notes in Mathematics, Vol. 523, Springer-Verlag, Berlin.

Albevario, S. A., and Hoegh-Krohn, R. J. (1979). Feynman path integrals and the corresponding method of stationary phase, in *Feynman Path Integrals*, S. A. Albevario *et al.*, eds., Lecture Notes in Physics, Vol. 108, Springer-Verlag, Berlin, pp. 3-57.

Cameron, R. H. (1960). A family of integrals serving to connect the Wiener and Feynman integrals, *Journal of Mathematical Physics*, **39**, 126-140.

Cameron, R. H. (1962-63). The Ilstow and Feynman integrals, *Journal d'Analyse Mathematique*, **10**, 287-361.

Cameron, R. H. (1968). Approximation to certain Feynman integrals, *Journal d'Analyse Mathematique*, **21**, 337-371.

Cheng, K. S. (1972). Quantization of a general dynamical system by Feynman's path integration formulation, *Journal of Mathematical Physics*, **13**, 1723-1725.

Cheng, K. S. (1973). Quantization by Feynman's path integration method, *Journal of Mathematical Physics*, **14**, 980-981.

Choquet-Bruhart, Y., DeWitt-Morette, C., and Dillard-Bleick, M. (1982). *Analysis, Manifolds and Physics*, ref. ed., North-Holland, Amsterdam.

Cohen, H. A., and Shaharir, b. M. Z. (1974). The action principle in quantum mechanics, *International Journal of Theoretical Physics*, **11**, 289-302.

DeWitt, B. (1957). Dynamical theory in curved space. I. A review of the classical and quantum action principles, *Review of Modern Physics*, **29**, 377-397.

DeWitt-Morette, C. (1972). Feynman path integrals, definition without limiting procedure, *Communications in Mathematical Physics*, **28**, 47-67.

DeWitt-Morette, C. (1974). Feynman path integrals. I. Linear and affine techniques; II. The Feynman-Green function, *Communications in Mathematical Physics*, **37**, 68-81.

DeWitt-Morette, C. (1976). The semiclassical expansion, *Annals of Physics*, **97**, 367-399; erratum, **101**, 682-683.

DeWitt-Morette, C. (1979). A reasonable method for computing path integrals on curved space, in *Feynman Path Integrals*, S. A. Albevario *et al.*, eds., Lecture Notes in Physics, Vol. 106, Springer-Verlag, Berlin, pp. 227-233.

DeWitt-Morette, C., Maheswari, A., and Nelson, B. (1979). Path integration in non-relativistic quantum mechanics, *Physics Reports*, **50**, 255-372.

DeWitt-Morette, C., Elworthy, K. D., Nelson, B. L., and Sammelson, G. S. (1980). A stochastic scheme for constructing solutions of Schrödinger equations, *Annales de l'Instut Henri Poincaré*, **A32**, 327-341.

Eliasson, H. I. (1967). Geometry of manifolds of maps, *Journal of Differential Geometry*, **1**, 169-194.

Elworthy, K. D. (1982). *Stochastic Differential Equations on Manifolds*, London Mathematical Society Lecture Note Series 70, Cambridge University Press.

Elworthy, K. D., and Truman, A. (1981). Classical mechanics, the diffusion (heat) equation and the Schrödinger equation on Riemannian manifolds. *Journal of Mathematical Physics*, **22**, 2144-2166.

Exner, P. (1985). *Open Quantum Systems and Feynman Integrals*, Reidel, Dordrecht.
Feynman, R. P. (1948). Space-time approach to non-relativistic quantum mechanics, *Review of Modern Physics*, **20**, 367-387.
Feynman, R. P., and Hibbs, A. R. (1965). *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York.
Glimm, J., and Jaffe, A. (1981). *Quantum Physics: A Functional Integral Point of View*, Springer-Verlag, Berlin.
Golubitsky, M., and Guillemin, V. (1973). *Stable Mappings and Their Singularities*, Springer-Verlag, Berlin.
Hormander, L. (1971). Fourier integral operators, *Acta Mathematica*, **127**, 79-183.
Ito, K. (1961). Wiener integral and Feynman integral, in *Proceedings Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 2, University of California Press, pp. 227-238.
Johnson, G. W. (1982). The equivalence of two approaches to Feynman integrals, *Journal of Mathematical Physics*, **23**, 2090-2096.
Kac, M. (1949). On distribution of certain Wiener functional, *Transactions of the American Mathematical Society*, **65**, 1-13.
Kac, M. (1959). *Probability and Related Topics in Physical Science*, Interscience, New York.
Kun Soo Chang, Johnson, G. W., and Skoug, D. L. (1984). Necessary and sufficient conditions for the Fresnel integrability of certain classes of functions, *Journal of the Korean Mathematical Society*, **21**, 21-29.
Lang, S. (1985). *Differential Manifolds*, Springer-Verlag, Berlin.
Lin, H. E., Lin, W. C., and Sugano, R. (1970). On velocity-dependent potentials in quantum mechanics, *Nuclear Physics*, **B16**, 431-449.
McLughlin, D. W., and Schulman, L. S. (1971). Path integral in curved spaces, *Journal of Mathematical Physics*, **12**, 2521-2524.
Nelson, E. (1964). Feynman integrals and the Schrödinger equation, *Journal of Mathematical Physics*, **5**, 332-343.
Reed, M., and Simon, B. (1975). *Method of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academic Press, New York.
Schulman, L. S. (1981). *Techniques and Applications of Path Integration*, Wiley, New York.
Shaharir, b. M. Z. (1986a). On complex normal distributions, *Journal Sains Malaysiana*, to appear.
Shaharir, b. M. Z. (1986b). The non-uniqueness of Schrödinger equation derived via stochastic mechanics, *Letters in Mathematical Physics*, submitted.
Spivak, M. (1979). *A Comprehensive Introduction to Differential Geometry*, Vol. II, Publish or Perish.
Streit, L., and Hida, T. (1983). Generalized Brownian functional and the Feynman integral, *Stochastic Processes and Their Applications*, **16**, 55-69.
Tarski, J. (1979). Feynman-type integrals defined in terms of general cylindrical approximations, in *Feynman Path Integrals*, S. A. Albevario *et al.*, eds., Lecture Notes in Physics, Vol. 106, Springer-Verlag, Berlin, pp. 254-279.
Truman, A. (1976). Feynman path integrals and quantum mechanics as $\hbar \to 0$, *Journal of Mathematical Physics*, **17**, 1852-1862.
Truman, A. (1977). Classical mechanics, the diffusion (heat) equation and the Schrödinger equation, *Journal of Mathematical Physics*, **18**, 2308-2315.
Truman, A. (1978). The Feynman maps and the Wiener integral, *Journal of Mathematical Physics*, **19**, 1742-1750.
Truman, A. (1979). The polygonal path formulation of the Feynman path integrals, in *Feynman Path Integrals*, S. A. Albevario *et al.*, eds., Lecture Notes in Physics, Vol. 106, Springer-Verlag, Berlin, pp. 73-102.